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Matching Solutions for Unsteady Conduction in Simple Bodies with Surface Heat Fluxes

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Nomenclature

k = thermal conductivity
L = slab semithickness or radius of cylinder and sphere
q_s = uniform surface heat flux
T = temperature
t = temporal variable
X, X' = dimensionless x, x'; x/L, x'/L
x = space variable measured from the center
x' = space variable measured from the surface, $x + x' = L$
α = thermal diffusivity
τ = dimensionless t, $\alpha t/L^2$
ϕ = dimensionless T, $k(T - T_i)/q_sL$

Introduction

AMONG the arsenal of analytical methods of solution for the treatment of unsteady state heat conduction in simple coordinate systems, the method of separation of variables is by far the most commonly adopted in literature on conduction heat transfer. However, its step-by-step implementation presents inconveniences on some occasions, especially when the partial differential equation and/or the boundary conditions involve nonhomogeneities. Under these circumstances utilization of other solution techniques are more appropriate and sometimes become indispensable. In principle, the method of separation of variables leads to an exact infinite series that is conveniently simplified to an approximate one-term of series solution for practical applications. The latter provides the so-called long-time solution and serves to generate different versions of temperature–time charts that are applicable to the cooling of simple bodies with convective surfaces.

The central objective of the present study is twofold. First, to establish a definite region of validity of the venerable one-term of series solution for simple bodies exposed to a uniform surface heat flux. Second, to supplement this simplified solution with an optional technique having physical roots, i.e., the semi-infinite body solution. The latter, supposedly may permit a faster and more precise calculation for the temperature response of simple bodies for short times. Evaluation of the compact semi-infinite body solution involving one or two terms only may be carried out by hand. To the author's knowledge the literature does not contain information pertinent to the previously cited objectives.

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Mathematical Formulation

At $t = 0$, q_s is suddenly applied at the surface $x = L$ of a simple body (slab, cylinder, sphere) with constant thermophysical properties. These bodies possess a uniform temperature T_i for $t < 0$. The one-dimensional, transient heat conduction equation, along with the initial and boundary conditions, are

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial X^2} + \frac{c}{X}\frac{\partial \phi}{\partial X} \tag{1}$$

where the parameter c becomes 0 (slab), 1 (cylinder), and 2 (sphere).

$$\phi(X, 0) = 0 \tag{2a}$$

$$\frac{\partial \phi}{\partial X}(0, \tau) = 0 \tag{2b}$$

$$\frac{\partial \phi}{\partial X}(1, \tau) = 1 \tag{2c}$$

Method of Separation of Variables

This method constitutes the baseline solution. As is customarily done, the superposition of functions

$$\phi(X, \tau) = f_1(\tau) + f_2(X) + f_3(X, \tau) \tag{3}$$

can eliminate the ensuing difficulty arising from the nonhomogeneous boundary condition of Eq. (2c). Thus, the series solutions, $\phi(X, \tau)$, and the characteristic values μ_n are found in Luikov[1]:

Slab:

$$\phi(X, \tau) = \tau - \frac{1}{2}\left(\frac{1}{3} - X^2\right) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\mu_n^2}\cos(\mu_n X)e^{-\mu_n^2\tau} \tag{4}$$

where $\mu_n = n\pi$, $n = 1, 2, 3 \ldots$

Cylinder:

$$\phi(X, \tau) = 2\tau - \frac{1}{4}(1 - 2X^2) - \sum_{n=1}^{\infty}\frac{2}{\mu_n^2 J_0(\mu_n)}J_0(\mu_n X)e^{-\mu_n^2\tau} \tag{5}$$

where $J_0'(\mu_n) = J_1(\mu_n)$, $n = 1, 2, 3 \ldots$

Sphere:

$$\phi(X, \tau) = 3\tau - \frac{1}{2}\left(\frac{3}{5} - X^2\right) - 2\sum_{n=1}^{\infty}\frac{1}{\mu_n \cos\mu_n}\frac{\sin(\mu_n X)}{\mu_n X}e^{-\mu_n^2\tau} \tag{6}$$

where $\tan\mu_n = \mu_n$, $n = 1, 2, 3 \ldots$

It is widely known that the pressing characteristic of the preceding infinite series is that they tend to converge rapidly for long times. The mere presence of only one term in the series suffices to produce acceptable results. On the contrary, these infinite series show severe divergence patterns for short times even when many terms are present. The behavior becomes so abnormal that the numerical evaluation of the one-term of series solution does not meet the initial condition [Eq. (2a)]. In fact, for very short times the local temperatures consistently overpredict the initial condition in contraposition with the physics of the problem. This contrasting situation poses a clear disadvantage for the calculation of problems in thermal engineering.

As a point of reference, Grigull and Sandner[2] have pointed out that the rigorous validity of the one-term of series solution for uniform surface temperature is articulated with a critical dimensionless time, τ_{cr} = 0.24, 0.21, and 0.18 for slabs, cylinders, and spheres, respectively, having a 1% accuracy. Despite its remarkable importance in thermal engineering, the region of validity for the one-term of series solution connected with the uniform heat flux boundary condition has not been established so far.

Solution for a Semi-Infinite Body

The temperature response for the initial stages of heating, $T(x', t)$, can be taken from the textbook by Luikov[1]:

$$T(x', t) = T_i + (q_s/k)\{(4\alpha t/\pi)\exp(-x'^2/4\alpha t) - x' \operatorname{erfc}[x'/(4\alpha t)^{1/2}]\} \tag{7}$$

where x' is measured from the surface of the body inward. By virtue of Eq. (7), the temperature–time variation of the surface and the center are given, respectively, by the one- and two-term equations:

$$\phi(X' = 0, \tau) = 2(\tau/\pi)^{1/2} \tag{8}$$

$$\phi(X' = 1, \tau) = [2(\tau/\pi)^{1/2}e^{-(1/4\tau)} - \operatorname{erfc}(1/2\tau^{1/2})] \tag{9}$$

Discussion of Results

The question that needs to be answered now is connected with the accuracy provided by the semi-infinite body solution for short times and the one-term of series solutions for long times, when contrasted with the exact infinite series solutions. The latter have been generated by evaluating Eqs. (4–6) with software on symbolic mathematics. About 1300 terms have been retained in the early stages of heating, reducing the number of terms gradually with adequate controls as time progresses. Even more important is the possible blending of the two approximate solutions and the establishment of two subregions of validity of each, in other words the actual sizes of the time intervals $\Delta\tau$ where they respectively hold. The error $\varepsilon = \phi_{\text{approx}} - \phi_{\text{exact}}$ is preferred here because of the small numbers that intervene in the calculations.

Slab: Beginning with the semi-infinite body solution, it can be observed in Fig. 1a that the surface and center temperatures can be predicted accurately up to τ = 0.3 and 0.08, respectively. Thereafter, this approximate physical solution deteriorates and underpredicts both local temperatures producing errors that increase prominently with time.

The outcome of the one-term of series solution for the surface and center temperatures will now be examined. To harmonize with the intrinsic convergence characteristic of the truncated series, it is beneficial to initiate the discussion with a large value of τ, and move backwards, decreasing τ slowly. It may be observed in Fig. 1a that for $\tau > 0.08$ the errors for the surface and center are imperceptible. For $\tau < 0.08$ the solution becomes invalid and the local temperatures cannot be predicted reasonably. In addition, inspection of Fig. 1a also reveals that in the intermediate subinterval, $0.05 < \tau < 0.3$, the three curves for the surface temperature are practically superimposed and the corresponding deviations are imperceptible at the scale of the graph. Also, the center temperature detaches from the initial condition $\phi = 0$, at a late time, approximately $\tau = 0.08$.

Cylinder: From the plots in Fig. 1b it can be inferred that the semi-infinite body solution produced results for the surface and center temperatures of good accuracy up to τ = 0.01 and 0.06, respectively. Beyond these times, the approximate physical solution breaks down and underpredicts the local temper-

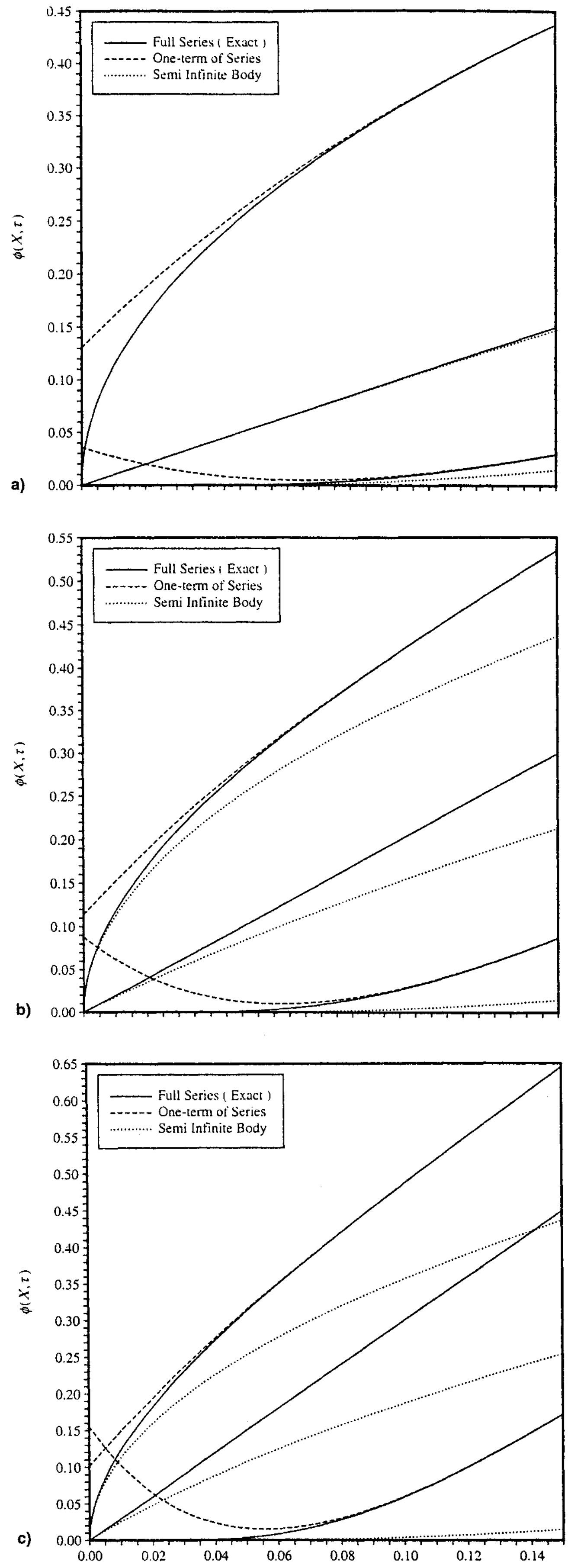

Fig. 1 Comparison of surface and center temperatures for a) slab, b) cylinder, and c) sphere.

atures, yielding errors that increase markedly with time. Consequently, the solution behaves well in the immediate vicinity of the origin only. For $\tau > 0.01$ it is useless because its performance vanishes with the curvature effects in the cylinder.

The behavior of the surface and center temperatures as predicted by the one-term of series solution will be addressed now. Here again, the discussion will begin with a large value of τ, and move backwards towards the origin, slowly decreasing the time slowly. Whenever $\tau > 0.06$ and $\tau > 0.08$, Fig. 1b reveals that the errors for the surface and center temperatures are small. For the subintervals associated with times less than those just indicated, use of the solution yields erroneous results for the local temperatures. In addition, inspection of Fig. 1b shows evidence that the center temperature detaches from the initial condition, $\phi = 0$, at an early time of approximately $\tau = 0.06$.

Sphere: Fig. 1c displays the temperature evolution of the surface and center temperatures supplied by the two different procedures. Obviously, the semi-infinite body solution predicts the surface temperature accurately for very diminutive times, i.e., $\tau < 0.005$. Calculation of the center temperature loses significance because the detachment from the initial condition $\phi = 0$ does not take place until approximately $\tau = 0.05$. This approximate physical solution behaves well in the immediate vicinity of the origin only. The solution's precision disappears with vigorous curvature effects in the sphere and for $\tau > 0.005$ its use is counterproductive.

Next, the surface and center temperatures as predicted by the one-term of series solution will be discussed. Here again, it is advantageous to initiate the discussion with a large value of τ, and move backwards towards the origin decreasing the time slowly. Figure 1c reveals that the errors for the surface and center are negligible in the time subintervals, $\tau > 0.04$ and $\tau > 0.08$, respectively. For the subintervals associated with times less than those indicated, the solution yields erroneous results for the local temperatures. In addition, inspection of Fig. 1c reflects that the center temperature detaches from the initial condition $\phi = 0$ at a very early time, approximately $\tau = 0.05$.

In summary, the semi-infinite body solution has a regular behavior for short times and does not overpredict the initial condition as τ approaches zero for the three configurations in question. On the contrary, the one-term of series solution as expected does overpredict markedly the initial condition as τ tends to zero.

Conclusions

A critical dimensionless time $\tau_{cr} = 0.1$ seems to be a realistic borderline for the validity of the one-term of series solution and the semi-infinite body solution in estimating the surface and center temperatures of slabs. A perfect blending of these two solutions, one the inner solution and the other the outer solution, can be achieved, thereby producing a map of solutions for this geometry. Unfortunately, for the cylinder and the sphere, the semi-infinite body solution can be employed safely up to minuscule dimensionless times and obviously the blending of the two solutions does not materialize. These statements reinforce the fact that the semi-infinite body solution is a restricted approximate route that does work for the planar slab, but fails for the cylinder and the sphere, because of the presence of moderate and strong curvature effects. This study clarified in a convincing manner a generalized belief that the semi-infinite body solution could be applied indiscriminatory to all geometries at short times, irrespective of their shapes.

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Combined Conduction and Nongray Radiation Heat Transfer in Carbon Dioxide

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Introduction

THE problem of heat transfer by simultaneous conduction and radiation in an absorbing–emitting medium has been a long-standing topic in high-temperature thermal engineering, and a number of studies have been made to quantitatively understand the combined effects of radiation and conduction.

The majority of earlier publications concerning this subject has been theoretical investigations based on the gray-gas approximation[1-3] to the radiative properties of infrared (IR) gases, which is today, considered to be incorrect: the gray-gas model shows a much greater pressure dependence and overestimates the effect of radiation.

Unlike these studies, several researchers theoretically analyzed the interaction between radiation and conduction utilizing the real-gas models such as the picket-fence models,[4-6] the exponential wideband model,[7] and the weighted-sum-of-gray-gases model,[8] but there exist a quite limited number of experimental studies. Only Schimmel et al.[9] and Novotny and Olsofka[10] have experimentally investigated the ability of the used gas model, i.e., the exponential wideband model, to predict the interaction of gaseous radiation with conduction in molecular gases and concluded that the exponential wideband model is correct for estimating local interaction effect in a radiating–conducting layer.

However, it should be noted that the exponential wideband model was originally assumed to be applicable to isothermal cases, and thus, to extend this model to nonisothermal cases, the Curtis–Godson approximation must be additionally introduced. Moreover, the effect of nonblack walls could not be taken into account in this model.

To overcome these defects, wideband spectral models for the absorption coefficients of IR gases are needed. The two-parameter wideband spectral model[11] developed recently by the author seems to be a promising means that can be used for this purpose, but the applicability of this model to coupled heat transfer problems in participating media has not yet been discussed.

In this Note first, the problem of combined conduction and nongray radiation heat transfer through a plane–parallel layer of carbon dioxide is theoretically formulated utilizing the two-parameter wideband spectral model and, then, to examine the validity of this model in predicting combined heat transfer in a radiating–conducting medium, numerical computations are made under conditions corresponding to the experiments by Mori and Kurosaki[3] and Schimmel et al.,[9] and the obtained theoretical results are compared with the experimental ones. Finally, the simple adding method[12] previously developed for approximately evaluating the total heat flux in gray media is extended to cover nongray situations and its adequacy is addressed.

Governing Equations

The present analyses are based on the following assumptions:

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